Asymptotic dimension, distributed algorithms, and local graph concepts

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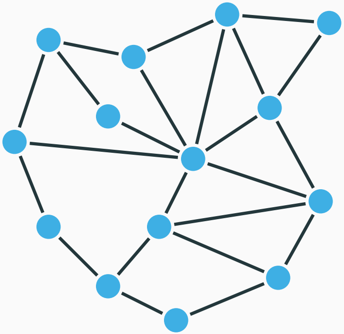
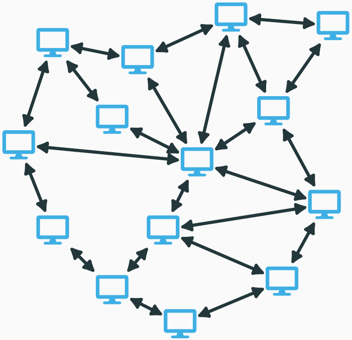
Distributed algorithms

Centralized view



Distributed view



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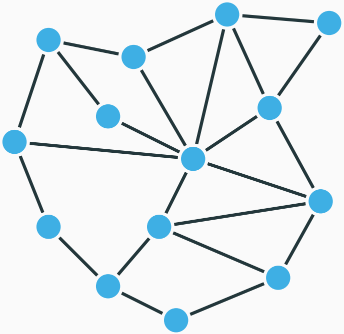
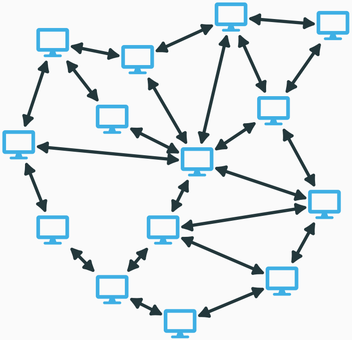


Focused on computing

Distributed view

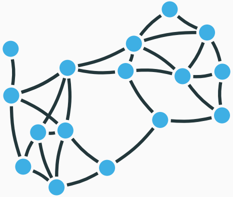


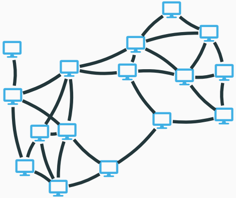
Focused on communication

The LOCAL model

The LOCAL model

graph





network

vertices

+

edges

UID UID

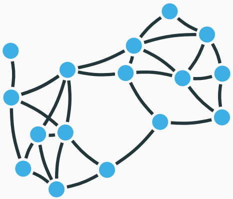
computers

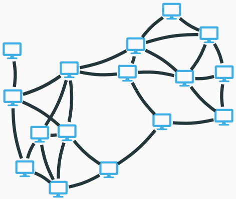
+

connections

The LOCAL model

graph





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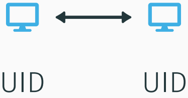


computers

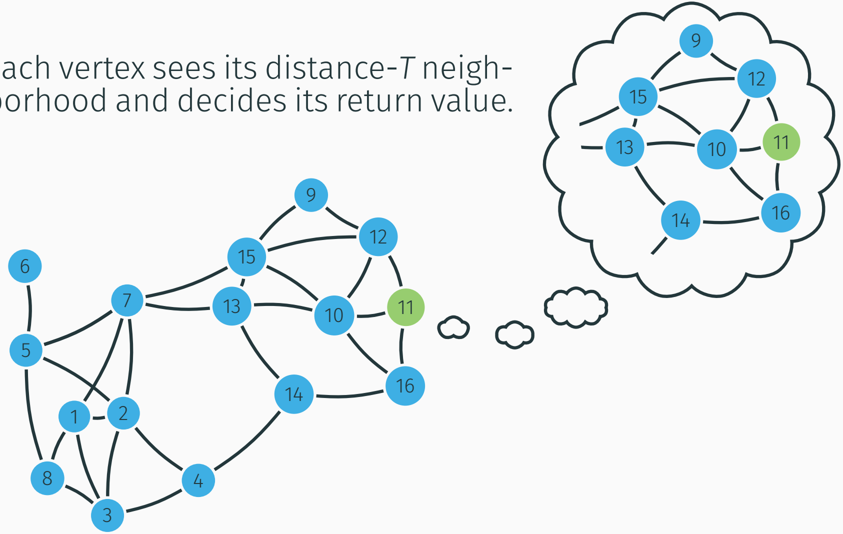
+

connections

The network is also the input graph!

Running time T

Each vertex sees its distance-T neighborhood and decides its return value.



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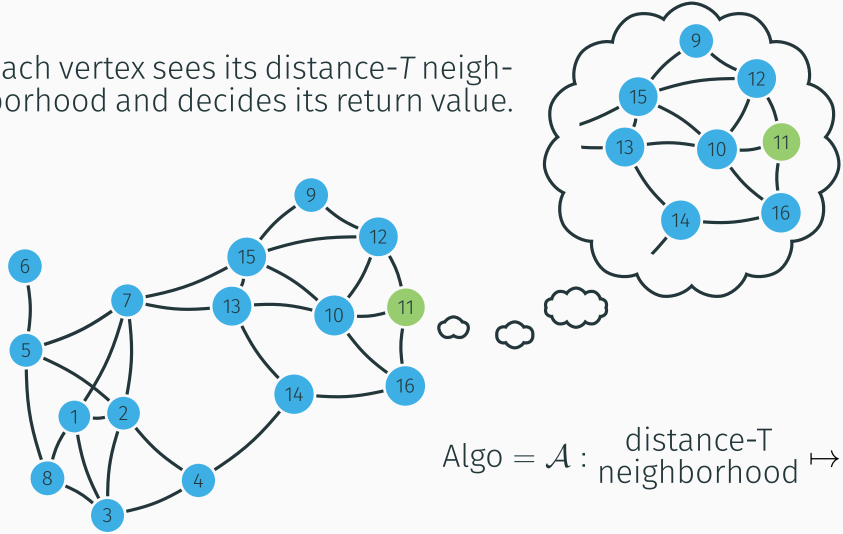
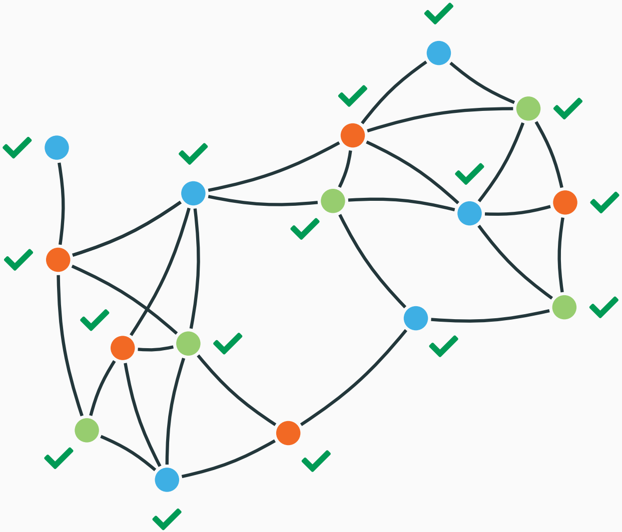


$$\text{Algo} = \mathcal{A} : \begin{matrix}\text{distance-T} \\ \text{neighborhood}\end{matrix} \mapsto \begin{matrix}\text{local} \\ \text{return value}\end{matrix}$$

An example: 3-coloring



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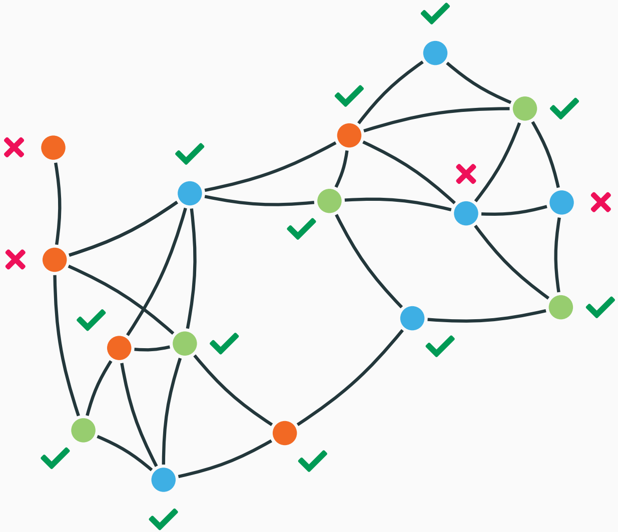
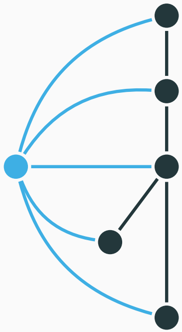


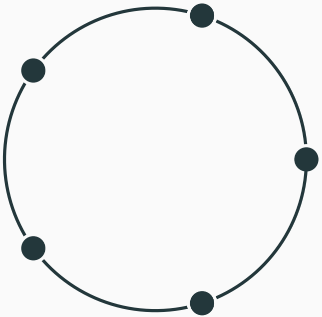
Complexity differences between LOCAL and centralized

Maximum Independent Set
when ∃ universal vertex



Easy in LOCAL
Hard in centralized

Detecting Cycles



Hard in LOCAL
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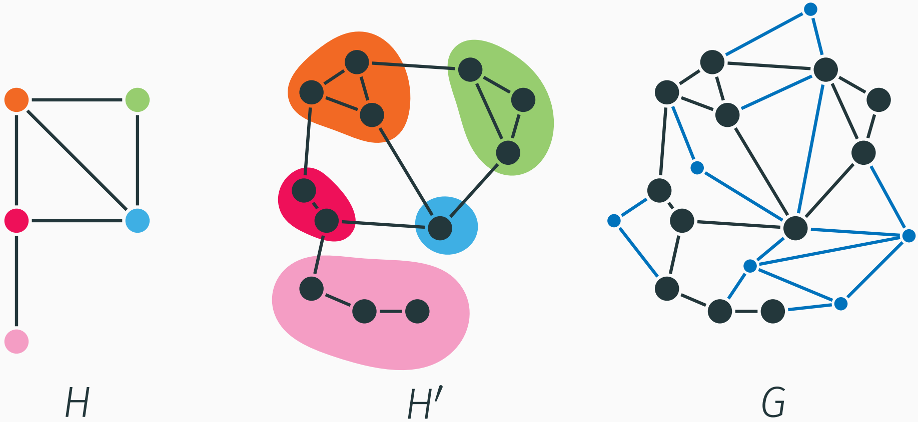
Graph minors



H is a minor of G

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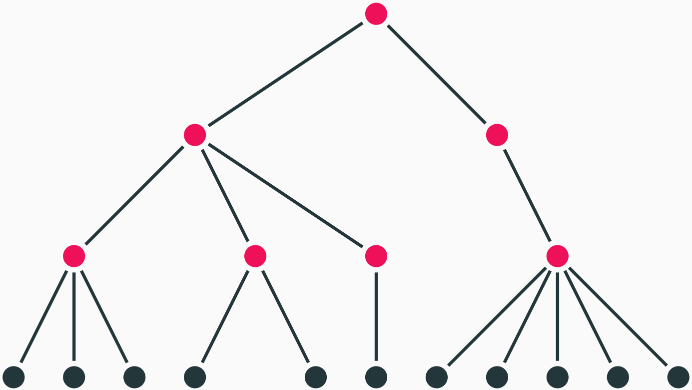
- $K_{2,t}$-minor-free graphs
  - $\mathcal{O}(1)$-approximation

Example 1: trees



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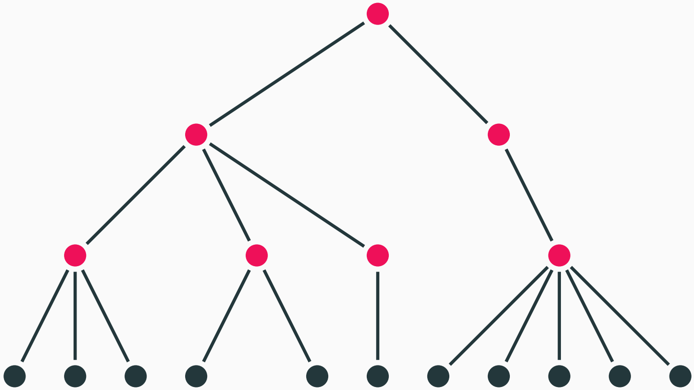
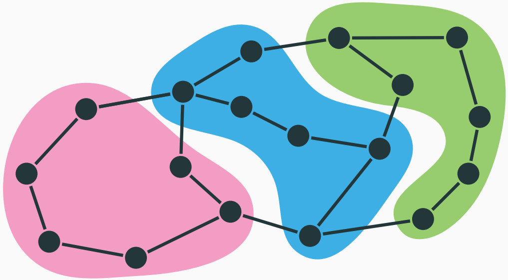


Theorem

$|\{v \in V(T) \mid \mathrm{d}(v) \geq 2\}| \leq 3 \cdot \mathrm{MDS}(T)$

Example 2: high girth graphs



Reuse the analysis of trees?

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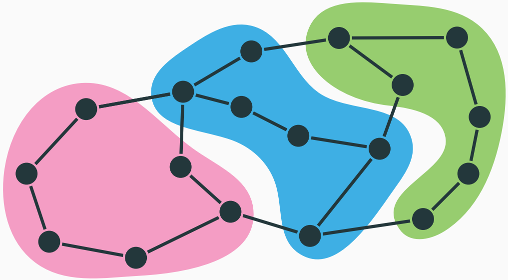

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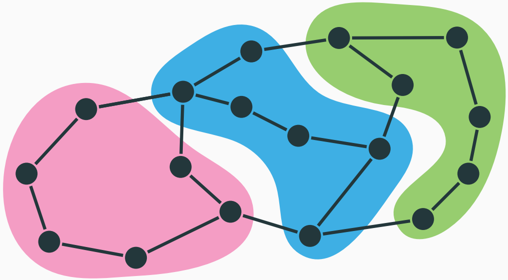

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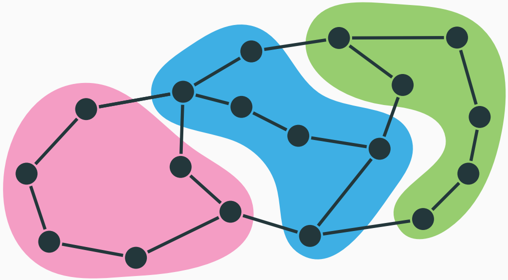

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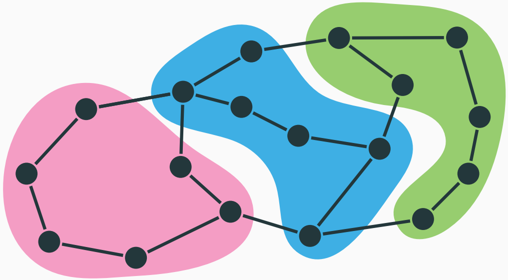

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- Every vertex is in a patato
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- Spacing between same colors $\implies$ in parallel
- Finite number of colors

Asymptotic dimension

Asymptotic dimension $= d$ if $\forall r, \exists C_1, C_2, \ldots, C_{d+1} \subseteq \mathcal{P}(V(G)), \exists f : \mathbb{N} \to \mathbb{N}$ such that

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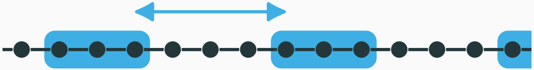

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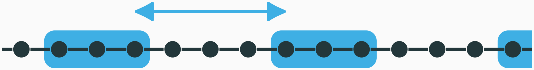
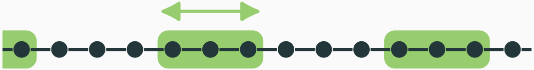
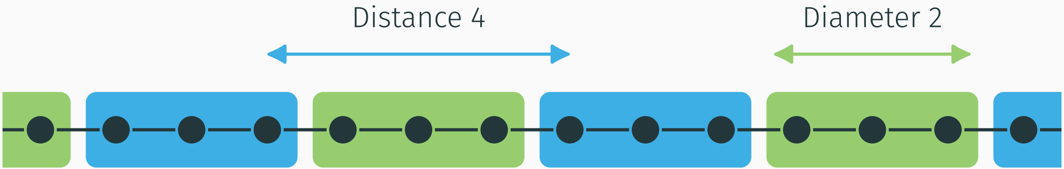


- **Boundedness:** $\forall B \in C_i, \mathrm{diam}_G(B) \leq f(r)$

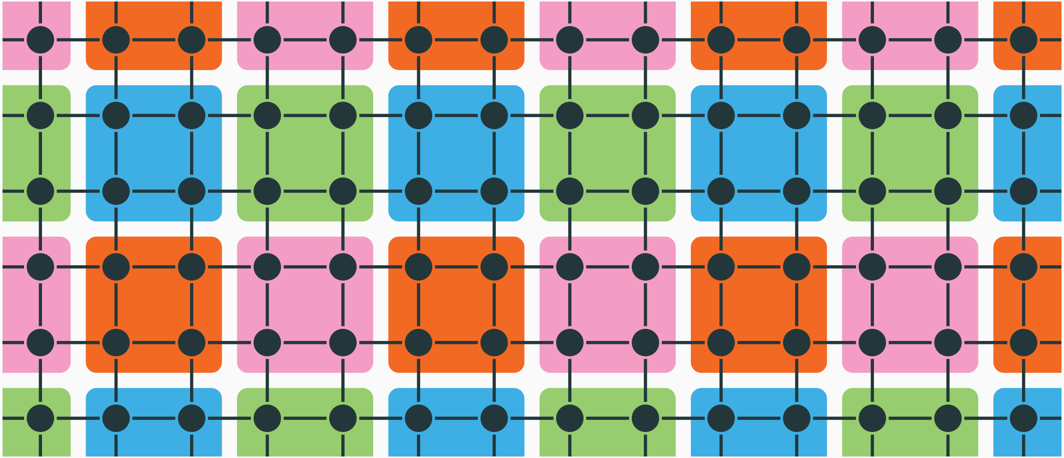


Example 1: the path



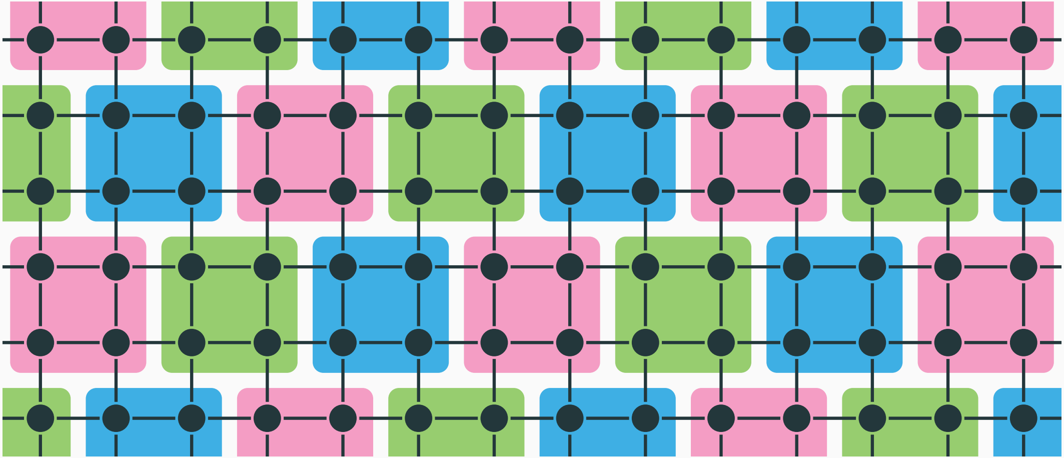
Dimension $= 1$

Example 2: the grid – try 1



Dimension ≤ 3

Example 2: the grid – try 2



$$\text{Dimension} = 2!$$

Asymptotic dimension and graph minors

Theorem (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, Scott, 2020)

Every class forbidding a minor has asymptotic dimension ≤ 2.

Application: distributed algorithms

How to use graph theory in distributed algorithms ?

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How to use graph theory in distributed algorithms ?

Global concept

$\downarrow$

Local concept

How to use graph theory in distributed algorithms ?

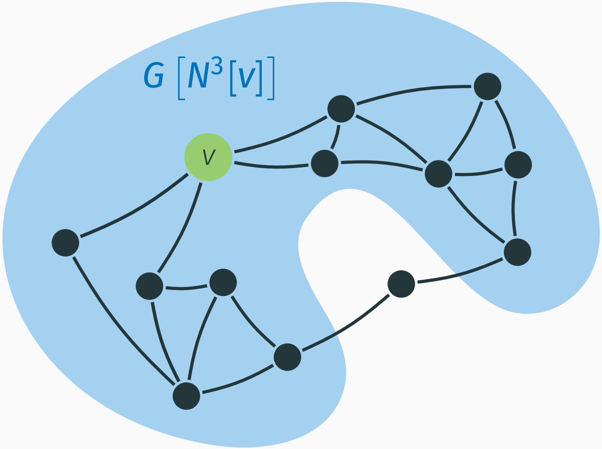
Global concept

↓

Local concept

Definition

v is a *r-local cutvertex* if v is a cutvertex of $G[N^r[v]]$.



Cutvertices locaux

Theorem

For every graph G, #cutvertices $\leq 3\,\mathrm{MDS}(G)$.

$$\Downarrow$$

Theorem

Let $\mathcal{C}$ be of asymptotic dimension d.

Then $\forall r \geq r(\mathcal{C})$, #***r*-local** *cutvertices* $\leq 3(\boldsymbol{d}+1)\,\mathrm{MDS}(G)$.

Cutvertices locaux

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For every graph G and $\boldsymbol{S} \subseteq \boldsymbol{V(G)}$, $\#\text{cutvertices} \in \boldsymbol{S} \leq 3\,\text{MDS}(G, \boldsymbol{N}[S])$.

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Dimension d, function f: sets $C_1, C_2, \ldots, C_{d+1}$ for $r = 5$.

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Let S be of weak-diameter $f(5)$.

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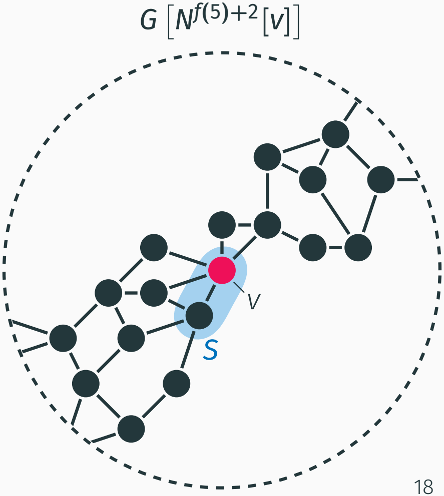

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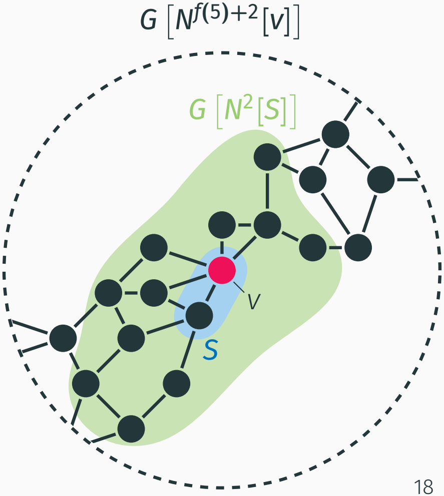
Claim: $N^2[S] \subseteq N^{f(5)+2}[v]$.



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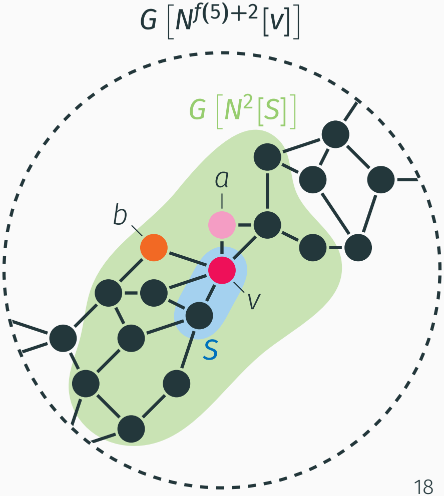

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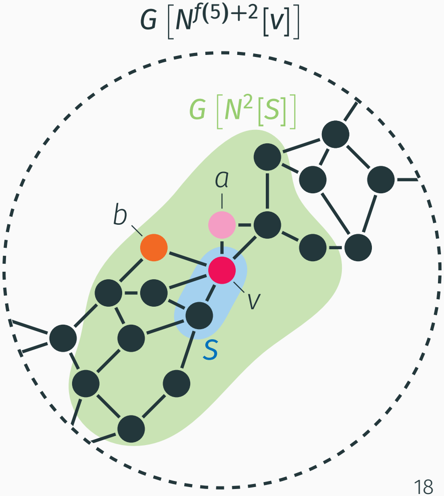

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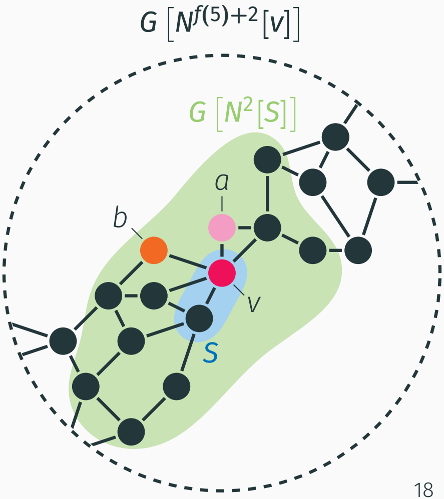

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$(N^2[S]$ are disjoint)

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Applications: local 2-cuts

Theorem

Let $\mathcal{C}$ of asymptotic dimension d.

Then $\forall r \geq r(\mathcal{C})$, $\#$vertices $\in$ r-local 2-cut $\leq 8(d+1)\,\mathrm{MVC}(G)$.

Theorem

If there exists a LOCAL algorithm:

- α*-approximation of MDS*
- *on* $\mathcal{C}$
- *in constant time r*
- *+ technical condition*

Applications: approximations on locally-$\mathcal{C}$ classes

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and if $\mathcal{D}$ *is:*

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Then there exists a LOCAL $\alpha(d+1)$*-approximation of MDS on* $\mathcal{D}$ *in time r.*

Theorem

On graphs without the minor $K_{2,t}$, there exists an $\mathcal{O}(1)$-approximation (where the constant is ***independant of t****) of Minimum Dominating Set in the LOCAL model, in $f(t)$ rounds.*

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Previous bound on $K_{3,t}$-minor-free graphs: $(2+\varepsilon)\cdot(t+4)$ in $g(\varepsilon, t)$ rounds (Heydt, Kublenz, Ossona de Mendez, Siebertz, Vigny 2022).

Conclusion and perspectives

Recap: 2 steps to go from global to local

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$$\text{Asymptotic dimension} \rightarrow \text{set } S \text{ of small diameter}$$
$$\text{local cutvertex} \in S \implies \text{cutvertex in } G[S]$$

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- Without minor $H \rightarrow$ local $\mathcal{O}(\mathrm{pw}(H))$-approximation in constant time ?

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$$\forall G, \forall S \subseteq V(G), |\{\text{cutvertices}\} \cap S| \leq 3 \cdot MDS(G, N[S])$$

- Apply asymptotic dimension

Asymptotic dimension $\rightarrow$ set S of small diameter

local cutvertex $\in S \implies$ cutvertex in $G[S]$

Union bound to conclude 😁

? Without minor $H \rightarrow$ local $\mathcal{O}(\mathrm{pw}(H))$-approximation in constant time ?

☻ Thanks! ☻